Lecture 6, 10/10/12

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Matrix Algebra

- Matrix / matrix multiplication $A \cdot B$
- Matrix / number multiplication $a \cdot A$
- Matrix / matrix addition and subtraction. $A \pm B$
- Matrix division (maybe on Wed.)

$$A \cdot \vec{x} = \vec{b} \longrightarrow \vec{x} = A^{-1} \cdot \vec{b}$$

Recap of Inverse

- Only square matrices have an inverse.
- But, not all square matrices have them.
  - A square matrix with no inverse is called ***singular***.
  - An invertible matrix is called ***nonsingular.***

Computing the inverse

- Let A be an nxn square matrix.
  - Form the augmented matrix $\left[\ A \mid \mathbb{I}\ \right]$
  - Then row reduce to $\left[\ \mathbb{I} \mid A^{-1}\ \right]$

Elementary Matrices and the inverse

- An ***elementary matrix*** is a matrix that encodes an elementary row operation.

Row Swapping

- Let $E_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- Then $E_s \cdot A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Similarly

- This matrix swaps the second and third rows.

$$E_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Row multiplication

- This matrix multiplies the second row by 'c'.

$$E_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ c \cdot a_{21} & c \cdot a_{22} & c \cdot a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Combination of Rows

- This matrix replaces row 2 with b*R1 + c*R2.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b \cdot a_{11} + c \cdot a_{21} & b \cdot a_{12} + c \cdot a_{22} & b \cdot a_{13} + c \cdot a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Application

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$R_1 \to .5R1 \qquad R_2 \to -3R_1 + R_2 \qquad R_3 \to R_1 + R_3$$

Application Continued

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_3 \cdot E_2 \cdot E_1 \cdot A \sim \begin{bmatrix} 1 & \square & \square \\ 0 & \square & \square \\ 0 & \square & \square \end{bmatrix}$$

Elementary Matrices and the inverse

- So, successively multiplying by elementary matrices is the same as performing Gaussian Elimination.
- It can thus be used to compute an inverse and to solve a system.

Properties of an inverse matrix.

- Let A and B be nonsingular (ie. invertible). Then $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- Proof

$$\begin{aligned} B^{-1} \cdot A^{-1} \cdot (A \cdot B) &= B^{-1} \cdot (A^{-1} \cdot A) \cdot B \\ &= B^{-1} \cdot \mathbb{I} \cdot B \\ &= \mathbb{I} \end{aligned}$$

Matrix Transpose

$$\text{Let } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\text{Then we define } A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

- Everything is flipped along the diagonal.

Note

- The transpose of a matrix is not the same size as the original.

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

Properties of Transpose

$$(A^T)^T = A$$
$$(a \cdot A)^T = a \cdot A^T$$
$$(A + B)^T = A^T + B^T$$
$$(A \cdot B)^T = B^T \cdot A^T$$

- These are direct to show by writing down matrices and performing the operation.

Terminology

- We say a matrix is ***symmetric*** if $A^T = A$
- Notice that while any matrix can have a transpose, only a square matrix can be symmetric.

Application

- Suppose that in a given year, 30% of married people get divorced and 20% of single people get married.
- Suppose you start with 10000 people of which 8000 are married and 2000 are single.
- How many people are married at the end of the first year.

Step 1

- Define you unknowns.

$$x_m^0 = \text{Number of married people next year}$$
$$x_s^0 = \text{Number of single people next year}$$

Step3

- Setup equations

$$x_m^1 = .7x_m^0 + .2x_s^0$$
$$x_s^1 = .3x_m^0 + .8x_s^0$$

$$\underset{\vec{x}^1}{\begin{bmatrix} x_m^1 \\ x_s^1 \end{bmatrix}} = \underset{A}{\begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}} \cdot \underset{\vec{x}^0}{\begin{bmatrix} x_m^0 \\ x_s^0 \end{bmatrix}}$$

Markov Chain

- This is referred to as a "Markov Process".
- The key feature is that what happens next year doesn't depend on what happened this year!
- The problem can be formulated as

$$\vec{x}^1 = A \cdot \vec{x}^0$$

What about at year 4?

- How many married / single people are there?

Now	Year 1	Year 2	Year 3	Year 4
$\vec{x}^0$	$\vec{x}^1$	$\vec{x}^2$	$\vec{x}^3$	$\vec{x}^4$
	$\vec{x}^1 = A \cdot \vec{x}^0$	$\vec{x}^2 = A \cdot \vec{x}^1$		$\vec{x}^4 = A \cdot \vec{x}^3$
		$\vec{x}^2 = A^2 \cdot \vec{x}^0$		$\vec{x}^4 = A^4 \cdot \vec{x}^0$

Markov Process

- Multiplication by A advances you forward one year in time. Multiplication by A 'n' times advances you forward 'n' years.

Applications

- Markov chains are used to describe:
  - Demographics
  - Disease transmission
  - Marketing effects